$$\Rightarrow \sqrt{\frac{2}{e}} \lim_{n\to\infty}\left(\frac{1}{1+1}\right)$$

(OR)

$$\Rightarrow \sqrt{\frac{2}{e}}\left(\frac{1}{2}\right)$$

$$L = \frac{1}{\sqrt{2e}}$$

Shivam Sharma

Third solution. Let $I_n := \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - \int_1^n \frac{dx}{\sqrt{x}} = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\left(\sqrt{n}-1\right)$ and $I := \lim\limits_{n\to\infty} I_n$.

For difference $I_n - I$ in **[1]** was proved inequality

$$\frac{1}{2\sqrt{n+1/6}} < I_n - I < \frac{1}{2\sqrt{n+1/6}}, n \in \mathbb{N} \tag{1}$$

Since $s_n + 2 = I_n$ then $I = s + 2$ and, therefore, **(1)** $\Longleftrightarrow$

$$\frac{1}{2\sqrt{n+1/6}} < s_n - s < \frac{1}{2\sqrt{n+1/6}}, n \in \mathbb{N} \tag{2}$$

Hence, by Squeeze Principle

$$\lim_{n\to\infty} \sqrt{n}\,(s_n - s) = \frac{1}{2}.$$

Also, noting that $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ and $(2n-1)!! = \frac{(2n-1)!}{2^n n!}$ we obtain

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\sqrt[n]{(2n-1)!}}{2\sqrt[n]{n!}} = \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!}}{n^2} \cdot \frac{n}{\sqrt[n]{n!}} =$$

$$= \frac{e}{2} \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!}}{n^2} = \frac{e}{2} \lim_{n\to\infty} \left(\frac{\sqrt[2n]{(2n-1)!}}{n}\right)^2 = \frac{e}{2}\left(\lim_{n\to\infty} \frac{\sqrt[2n]{(2n)!}}{2n} \cdot \frac{2}{\sqrt[2n]{2n}}\right)^2 =$$

$$= \frac{e}{2}\left(\lim_{n\to\infty} \frac{\sqrt[2n]{(2n)!}}{2n} \cdot \frac{2}{\lim\limits_{n\to\infty} \sqrt[2n]{2n}}\right)^2 = \frac{e}{2} \cdot \left(\frac{1}{e} \cdot 2\right)^2 = \frac{2}{e}.$$

Thus,

$$\lim_{n\to\infty}(s_n - s)\sqrt[2n]{(2n-1)!!} = \lim_{n\to\infty}\sqrt{n}\,(s_n - s)\cdot\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt{n}} =$$

$$= \lim_{n\to\infty}\sqrt{n}\,(s_n - s)\cdot\sqrt{\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}} = \frac{1}{2}\cdot\sqrt{\frac{2}{e}} = \frac{1}{\sqrt{2e}}.$$

1. A.Sîntamărian, Some inequalities regarding a generalization of Ioachimescu's constant, Journal of Mathematical Inequalities vol.4,n.3(2010), 413-421.

Arkady Alt

Fourth solution. We show that

$$\lim_{n\to\infty}\sqrt{n}(s_n - s) = \frac{1}{2}$$

Indeed by using Cesaro–Stolz theorem we get

$$\lim_{n\to\infty}\sqrt{n}(s_n - s) = \lim_{n\to\infty}\frac{s_n - s}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{s_{n+1} - s_n}{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}} =$$

$$\lim_{n\to\infty}\frac{-2\sqrt{n+1} + 2\sqrt{n} + \frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}} =$$

$$= \lim_{n\to\infty}\frac{(\sqrt{n}\sqrt{n+1})(-2\sqrt{n+1} + 2\sqrt{n} + \frac{1}{\sqrt{n+1}})}{\sqrt{n} - \sqrt{n+1}} =$$

$$= \lim_{n\to\infty}\frac{-2\sqrt{n}(n+1) + 2n\sqrt{n+1} + \sqrt{n}}{\sqrt{n} - \sqrt{n+1}} =$$

$$= \lim_{n\to\infty}\frac{-2\sqrt{n}(n+1) + 2n\sqrt{n}\sqrt{1+\frac{1}{n}} + \sqrt{n}}{\sqrt{n} - \sqrt{n}\sqrt{1+\frac{1}{n}}}$$

$$= \lim_{n\to\infty}\frac{-2\sqrt{n}(n+1) + 2n\sqrt{n}\left(1 + \frac{1}{2n} - \frac{1}{8n^2} + O(\frac{1}{n^3})\right) + \sqrt{n}}{\sqrt{n} - \sqrt{n}\left(1 + \frac{1}{2n} + O(\frac{1}{n^2})\right)} =$$

$$= \lim_{n\to\infty}\frac{-\frac{1}{4\sqrt{n}}}{-\frac{1}{2\sqrt{n}}} = \frac{1}{2}$$